JOURNAL OF APPROXIMATION THEORY **80**, 1–9 (1995)

Qualitative Korovkin-Type Theorems for R_F-Convergence

JOSE L. FERNANDEZ MUNIZ

Facultad de Matematica y Cibernetica, Havana University, Havana 4, Cuba

Communicated by Dany Leviatan

Received September 1, 1990; accepted in revised form September 23, 1993

In this paper we study sequences of linear operators which are "almost positive" outside sets of small Jordan measure. For them, we prove Korovkin-type theorems in terms of a modification of the R-convergence used previously by W. Dickmeis, H. Mevissen, R. J. Nessel, and E. Van Wickeren and the test families of functions which the author introduced in a previous paper. © 1995 Academic Press, Inc.

1. INTRODUCTION

Let $[a, b]$ be a p-dimensional interval of $\mathbb{R}^p$. By B (respectively R, C) we denote the space of bounded (respectively Riemann integrable, continuous) complex functions on $[a, b]$. By μ we denote the Jordan measure in $[a, b]$.

In [1], W. Dickmeis, H. Mevissen, R. J. Nessel, and E. Van Wickeren introduced the following definition.

A sequence (f_n) in B is said to be (Riemann) R-convergent to $f \in B$ if as $n \to \infty$ we have

(i) $\|f_n\|_\infty = O(1)$,

(ii) $\bar{\int} \sup_{k \geq n} |f_k - f| = o(1)$,

where $\bar{\int} g = \bar{\int} g(t)\,dt$ denotes the upper Riemann integral of g over $[a, b]$. For short, we write $R - \lim f_n = f$.

They proved that such convergence satisfies several important properties, in particular, that the following assertions are equivalent (see Proposition 2.2 of [1]):

(a) $R - \lim_n f_n = f$;

(b) (i) $\|f_n\|_\infty = O(1)$;

(ii) for every $\varepsilon > 0$, there exists a sequence of Jordan measurable sets (I_n) satisfying $I_{n+1} \subset I_n$, $n \in \mathbb{N}$, and $\lim_n \mu(I_n) = 0$ (we write $I_n \downarrow 0$)

0021-9045/95 $6.00

such that for every natural number n and each t in $I_n^c = [a,b] \setminus I_n$ then $|f_n(t) - f(t)| < \varepsilon$.

Using this concept of convergence, H. Mevissen, R. J. Nessel, and E. Van Wickeren obtained in [2] a direct generalization of the Bohman–Korovkin Theorem for positive linear operators considering the set of test functions 1, x, x^2 ($[a,b] \subset \mathbb{R}$). In [3, 4] different extensions of that result were given (in $\mathbb{R}^p$) for so-called R-sequences of linear operators (see Definition 1.3 below) and more general classes of test functions which separate points. Recently, M. Campiti extended the concept of R-convergence to certain topological spaces and also obtained Korovkin-type theorems working with sequences of (real) positive or contractive operators (see [5, 6]; these papers were brought to our attention by the referees).

Here, we consider complex sequences of linear operators which are "nearly" positive and prove Korovkin-type results using the test families introduced by us in [7]. To this end, we modify the notion of R-convergence as follows:

DEFINITION 1.1. We say that a sequence (f_n), $n \in \mathbb{N}$, in B is R_F-convergent to $f \in B$ if for every $\varepsilon > 0$, there exists $I_n \downarrow \varepsilon$ ($I_{n+1} \subset I_n$, $n \in \mathbb{N}$, $\lim \mu(I_n) < \varepsilon$) such that for every natural number n and every $t \in I_n^c$ we have that $|f_n(t) - f(t)| < \varepsilon$. In this case, we write $R_F - \lim f_n = f$.

As is to be expected, there is a close relationship between R and R_F-convergence. This is expressed in:

PROPOSITION 1.2. *Let (f_n), $n \in \mathbb{N}$, be a sequence of functions in B. Then for every $f \in B$ the following statements are equivalent:*

(a) $R - \lim f_n = f$.

(b) (i) $\|f_n\|_\infty = O(1)$;

(ii) $R_F - \lim f_n = f$.

Before stating the main result, we define precisely the type of linear operators which we study and recall the notion of test family which we use. From now on E denotes an arbitrary fixed subspace of R, which contains the constant functions.

DEFINITION 1.3. Let (L_n), $n \in \mathbb{N}$, be a sequence of linear operators from E into B. We say that (L_n), $n \in \mathbb{N}$, is an R_F-sequence in E if for every f in E with $\operatorname{Re} f \geq 0$, and for every $\varepsilon > 0$ there exists $I_n \downarrow \varepsilon$ such that for every $n \in \mathbb{N}$ and for every $t \in I_n^c$ then $\operatorname{Re} L_n(f,t) > -\varepsilon$.

Substituting $I_n \downarrow \varepsilon$ by $I_n \downarrow 0$ we get the R-sequences of linear operators used in [3] and [4] by M. Jimenez Pozo and E. Lopez Nunez.

DEFINITION 1.4. We say that $\{f_x\}$, $x \in [a,b]$, is a test family of functions in E if the following conditions hold:

(a) for every x in $[a,b]$, $f_x \in E$ and the function $(x,t) \to f_x(t)$ is continuous in $[a,b] \times [a,b]$;

(b) for every x in $[a,b]$, $\operatorname{Re} f_x(x) = 0$;

(c) for every t in $[a,b]$, $t \neq x$ and $\operatorname{Re} f_x(t) > 0$.

THEOREM 1.5. *Let $E \subset R$ be a linear subspace of R containing the constant functions and let $\{f_x\}$, $x \in [a,b]$, be a test family of functions in E. Let (L_n), $n \in \mathbb{N}$, be a sequence of linear operators from E into C. Then, $R_F - \lim L_N f = f$, for every $f \in E$, if and only if*:

(i) (L_n), $n \in \mathbb{N}$, *is an R_F-sequence*;

(ii) $R_F - \lim L_n 1 = 1$;

(iii) $R_F - \lim \operatorname{Re} L_n f_x = \operatorname{Re} f_x$, $x \in [a,b]$.

In Section 2, we prove Proposition 1.2 and some auxiliary results. Section 3 is devoted to the proof of Theorem 1.5. Some applications of this result are contained in Section 4.

Throughout the paper we maintain the notations introduced above.

2. AUXILIARY RESULTS

Proof of Proposition 1.2. It is clear that if $R - \lim f_n = f$, then $R_F - \lim f_n = f$.

Assume now that $R_F - \lim f_n = f$ and $\|f_n\|_\infty = O(1)$. Fix $M > 0$, such that $|f_n(t)| \le M$ and $|f(t)| \le M$ for t in $[a,b]$. Then for $\varepsilon > 0$, there exists $I_n \downarrow \varepsilon$ such that for t in I_n^c, $\sup_{k \ge n} |f_k(t) - f(t)| \le \varepsilon$. Thus

$$\overline{\int} \sup_{k \ge n} |f_k - f| \le \overline{\int_{I_n^c}} \sup_{k \ge n} |f_k - f| + \overline{\int_{I_n}} \sup_{k \ge n} |f_k - f|$$

$$\le \varepsilon\mu(I_n^c) + 2M\mu(I_n) \le \varepsilon\mu([a,b]) + 2M\mu(I_n).$$

Taking n sufficiently large so that $\mu(I_n) < \varepsilon$, then

$$\overline{\int} \sup_{k \ge n} |f_k - f| < [\mu([a.b]) + 2M]\varepsilon.$$

This completes the proof.

The next two results lead to the proof of Theorem 1.6.

THEOREM 2.1. *Let* (L_n), $n \in \mathbb{N}$, *be a sequence of linear operators, from* E *into* B. *Then,* $R_F - \lim L_n f = 0$ *for every* $f \in E$, *if and only if*:

(i) (L_n), $n \in \mathbb{N}$, *is an* R_F*-sequence*,

(ii) $R_F - \lim L_n 1 = 0$.

Proof. Obviously, properties (i) and (ii) are necessary.

Assume that (i) ad (ii) hold. Let $f \in E$, and set $M = \|f\|_\infty$. Take $\phi_1 = M - f$, $\phi_2 = M + f$; then $\phi_1, \phi_2 \in E$, and $\operatorname{Re} \phi_1 \geq 0$, $\operatorname{Re} \phi_2 \geq 0$.

Furthermore, if we fix $\varepsilon > 0$, by (i) there exists $I_n^1 \downarrow \varepsilon$ and $I_n^2 \downarrow \varepsilon$, such that for every $n \in \mathbb{N}$

$$\operatorname{Re} L_n(\phi_1, t) > -\varepsilon, \qquad t \in \left(I_n^1\right)^c,$$

$$\operatorname{Re} L_n(\Phi_2, t) > -\varepsilon, \qquad t \in \left(I_n^2\right)^c.$$

Now set $I_n^3 = I_n^1 \cup I_n^2$. Therefore, $I_n^3 \downarrow 2\varepsilon$ and for every $n \in N$,

$$\operatorname{Re} L_n(\phi_1, t) > -\varepsilon; \qquad \operatorname{Re} L_n(\phi_2, t) > -\varepsilon, t \in \left(I_n^3\right)^c.$$

By the linearity of the operators L_n we have

$$\begin{aligned} M \operatorname{Re} L_n(1, t) - \operatorname{Re} L_n(f, t) > -\varepsilon, \\ M \operatorname{Re} L_n(1, t) + \operatorname{Re} L_n(f, t) > -\varepsilon. \end{aligned} \tag{1}$$

On the other hand, by (ii) we know that there exists $I_n^4 \downarrow \varepsilon$, such that for every $n \in \mathbb{N}$

$$|ML_n(1, t)| < \varepsilon, \qquad t \in \left(I_n^4\right)^c. \tag{2}$$

Taking, $I_n^5 = I_n^3 \cup I_n^4$, we have that (1) and (2) are true simultaneously. Thus, $I_n^5 \downarrow 3\varepsilon$ and

$$-2 \in < \operatorname{Re} L_n(f, t) < 2\varepsilon, \qquad t \in \left(I_n^5\right)^c. \tag{3}$$

Putting $g = -if$ in (3), since $g \in B$ and $\operatorname{Re} g = \operatorname{Im} f$, there exists $I_n^6 \downarrow \varepsilon$, such that for every $n \in N$,

$$-2\varepsilon < \operatorname{Im} L_n(f, t) < 2\varepsilon, \qquad t \in \left(I_n^6\right)^c. \tag{4}$$

From (3) and (4), taking $I_n = I_n^5 \cup I_n^6$, we see that $I_n \downarrow 4\varepsilon$, and for every $n \in \mathbb{N}$,

$$|L_n(f, t)| < \varepsilon\sqrt{8}, \qquad t \in (I_n)^c.$$

The proof is complete.

With the aid of Theorem 2.1 we obtain:

THEOREM 2.2. *A sequence of linear operators* (L_n), $n \in \mathbb{N}$, *from* E *into* B, *satisfies the properties* $R_F - \lim L_n f = f$ *for every* $f \in E$, *if and only if there exists a sequence of linear operators* (B_n), $n \in \mathbb{N}$, *from* E *into* B, *such that for every* $f \in E$, $R_F - \lim B_n f = f$ *and*:

(i) $(L_n - B_n)$, $n \in \mathbb{N}$, *is an* R_F*-sequence*,

(ii) $R_F - \lim L_n 1 = 1$.

Proof. The necessity of (i) and (ii) is immediate. The sufficiency follows directly from Theorem 2.1 since the R_F-sequence $(L_n - B_n)$, $n \in \mathbb{N}$, satisfies

$$R_F - \lim (L_n - B_n)(1) = R_F - \lim L_n 1 - R_F - \lim B_n 1 = 0.$$

Thus, for every $f \in E$,

$$R_F - \lim (L_n - B_n)(f) = 0.$$

This completes the proof.

3. A KOROVKIN-TYPE THEOREM FOR R_F-SEQUENCES

As was pointed out above, this section is dedicated to:

Proof of Theorem 1.5. For the same reasons as above we concentrate on the sufficiency of the proof.

First we fix $\varepsilon > 0$ and f in E such that $\operatorname{Re} f \geq 0$. Set

$$\omega(f, x, \delta) = \sup\{|f(y) - f(z)| : y, z \in B(x, \delta) \cap [a, b]\},$$

where $B(x, \delta)$ is the open ball centered at x with radius $\delta > 0$, and

$$\omega(f, x) = \lim \omega(f, x, \delta), \qquad \delta \to 0^+.$$

Let

$$F(x, t) = \operatorname{Re} f(t) - \operatorname{Re} f(x) + \varepsilon + H \operatorname{Re} f_x(t),$$

where $F : [a, b] \times [a, b] \to \mathbb{R}$.

Because of the properties of the modulus of continuity we have that

$$\omega(F, (x, x)) \leq \omega(\operatorname{Re} f, x).$$

Denote $T = \{x \in [a, b] : \omega(\operatorname{Re} f, x) \geq \varepsilon\}$. The function f is R-integrable in $[a, b]$, therefore T is compact and $\mu(T) = 0$.

If $x \in [a,b] \setminus T$ then

$$\omega(F(x,x)) \leq \omega(\operatorname{Re} f, x) < \varepsilon,$$

hence there exists an open neighborhood V_x of (x,x), such that for $(y,z) \in V_x$ we obtain

$$F(y,z) \geq F(x,x) - \varepsilon = 0.$$

Take an elementary open set P such that $P \supset T$, and $\mu(P) < \varepsilon$. The set $A = \bigcup\{V_x : x \in [a,b] \setminus P\}$ is open. If $D = G \setminus A$, where $G = ([a,b] \setminus P) \times [a,b]$, then D is compact.

If $D \neq \varnothing$, then the continuous function $(x,t) \to \operatorname{Re} f_x(t)$ satisfies that for every $(x,t) \in D$, $\operatorname{Re} f_x(t) > 0$. Hence, there exists $m > 0$, such that for every $(x,t) \in D$ we have $\operatorname{Re} f_x(t) \geq m$. Then

$$\frac{\operatorname{Re} f(x) - \operatorname{Re} f(t) - \varepsilon}{\operatorname{Re} f_x(t)} \leq \frac{2\|f\| + \varepsilon}{m}.$$

Take $H \in \mathbb{R}$, such that

$$H \geq \frac{2\|f\| + \varepsilon}{m}.$$

Then

$$\operatorname{Re} \phi_x(t) \geq 0, \qquad (x,t) \in ([a,b] \setminus P) \times [a,b],$$

where ϕ_x is defined as

$$\phi_x = f - \operatorname{Re} f(x) + \varepsilon + H f_x.$$

Now, with $x \in [a,b] \setminus P$ fixed, by (i) of Theorem 1.6 we have that there exists $I_n^1(x) \downarrow \varepsilon$, such that for every $n \in \mathbb{N}$,

$$\operatorname{Re} L_n(\phi_x, t) > -\varepsilon, \qquad t \in I_n^1(x)^c.$$

Therefore,

$$\begin{aligned} \operatorname{Re} L_n(\phi_x, t) &= \operatorname{Re} L_n(f,t) - \operatorname{Re} f(x) \operatorname{Re} L_n(1,t) \\ &\quad + \varepsilon \operatorname{Re} L_n(1,t) + H \operatorname{Re} L_n(f_x, t) > -\varepsilon, \qquad (5) \end{aligned}$$

and so

$$\operatorname{Re} L_n(f,t) > \operatorname{Re} f(x) \operatorname{Re} L_n(1,t) - \varepsilon \operatorname{Re} L_n(1,t) - H \operatorname{Re}(f_x, t) - \varepsilon.$$

From (ii) and (iii) of Theorem 1.5 we obtain that there exists $I_n^2(x) \downarrow \varepsilon$, such that for every $n \in \mathbb{N}$

$$|H \operatorname{Re} L_n(f_x, t) - H \operatorname{Re} f_x(t)| < \varepsilon;$$
$$|(\operatorname{Re} f(x) - \varepsilon) \operatorname{Re} L_n(1, t) - (\operatorname{Re} f(x) - \varepsilon)| < \varepsilon. \tag{6}$$

Taking $I_n^3(x) = I_n^1(x) \cup I_n^2(x)$, from (5) and (6),

$$\operatorname{Re} L_n(f, t) > \operatorname{Re} f(x) - H \operatorname{Re} f_x(t) - 4\varepsilon, \qquad t \in (I_n^3(x))^c. \tag{7}$$

Let $K\colon [a, b] \times [a, b] - \mathbb{R}$ be the function defined by

$$K(x, t) = \operatorname{Re} L_n(f, t) - \operatorname{Re} f(x) + H \operatorname{Re} f_x(t).$$

Since the functions $\operatorname{Re} L_n(f, t)$ and $H \operatorname{Re} f_x(t)$ are continuous on $[a, b]$ and $[a, b] \times [a, b]$, respectively, we have the following inequalities for the modulus of continuity of the functions K and $\operatorname{Re} f$:

$$\begin{aligned} &\omega(K, (x, t)) \le \omega(\operatorname{Re} f, x), && (x, t) \in [a, b] \times [a, b] \\ &\omega(K, (x, t)) \le \omega(\operatorname{Re} f, x) < \varepsilon, && (x, t) \in ([a, b] \setminus T) \times [a, b]. \end{aligned} \tag{8}$$

Then, there exists a neighborhood $V(x, t)$ of (x, t) such that

$$|K(y, z) - K(w, u)| < \varepsilon, \qquad (y, z), (w, u) \in V(x, t).$$

Therefore,

$$K(y, z) > K(w, u) - \varepsilon.$$

If $u \in I_n^3(w)^c$, we know from (7) and the definition of K that $K(w, u) > -4\varepsilon$. Hence,

$$K(y, z) > -5\varepsilon. \tag{9}$$

Therefore,

$$\operatorname{Re} L_n(f, z) > \operatorname{Re} f(y) - H \operatorname{Re} f_y(z) - 5\varepsilon. \tag{10}$$

Take $I_n(x) = I_n^3(x) \cup P$, $n \in \mathbb{N}$. Then, $I_n(x) \downarrow 3\varepsilon$. Consider $V(x, t) = V_t(x) \times W_x(t)$, where $V_t(x)$ is a neighborhood of x that depends on t, and $W_x(t)$ is a neighborhood of t that depends on x.

The family of neighborhoods $\{W_x(t)\}$, $t \in [a, b]$, is an open covering of $[a, b]$. Thus there exist a finite set of points $\{t_1, t_2, \ldots, t_r\}$ such that the family $\{W_x(t_1), \ldots, W_x(t_r)\}$ is a covering of $[a, b] \supset I_n(x)^c$, $n \in N$.

Take $V(x) = \bigcap \{V_{t_i}(x)\colon i = 1, \ldots, r\}$. Then the family $\{V(x)\}$, $x \in [a, b] \setminus P$, is an open covering of $[a, b] \setminus P$, so there exists a set of points $\{x_1, x_2, \ldots, x_s\}$ such that $\{V(x_1), \ldots, V(x_s)\}$ is a finite covering of $[a, b] \setminus P$.

Let $I_n = \bigcup\{I_n(x_i): i = 1, \ldots, s\}$. We have that $I_n \downarrow 3s\varepsilon$. For every $x \in [a,b]\setminus P$, there exists $j \in \{1, \ldots, s\}$ such that $x \in V(x_j)$. If $t \in I_n^c$, then for all $i \in \{1, 2, \ldots, s\}$, $t \in I_n(x_i)^c$. In particular, $t \in I_n(x_j)^c$. Thus, there exists $h \in \{1, \ldots, r\}$ such that $t \in W_{x_j}(t_h)$.

We conclude that (x, t) and (x_j, t) are in $V_{t_h}(x_j) \times W_{x_j}(t_h)$. From (9) and (10), we obtain

$$\operatorname{Re} L_n(f, t) > \operatorname{Re} f(x) - H \operatorname{Re} f_x(t) - 5\varepsilon. \tag{11}$$

If $t \in I_n^c$, then $t \notin P$. Therefore, $t \in [a,b]\setminus P$, and we can take $t = x$ in (11). Then

$$\operatorname{Re} L_n(f, t) > \operatorname{Re}(f, t) - 5\varepsilon. \tag{12}$$

Denote by Id the identity operator from R in R. We can write (12) in the form

$$\operatorname{Re}((L_n - Id)f, t) > -5\varepsilon. \tag{13}$$

Since $(L_n - \mathrm{Id})$, $n \in \mathbb{N}$, is an R_F-sequence, by Theorem 2.2 the proof is complete.

4. Applications

In certain cases the test family may be reduced to a finite set of functions. In this direction an interesting application of Theorem 1.5 is:

COROLLARY 4.1. *Let E be a linear subspace of R, which contains the constant functions and is closed with respect to complex conjugation. Let $\{f_1, f_2, \ldots, f_p\}$ be a finite family of real functions, such that for every $i = 1, \ldots, p$, f_i is continuous in $[a,b]$, $f_i \in E$, and $f_i^2 \in E$. Assume also that $\{f_1, \ldots, f_p\}$ separates the points of $[a,b]$. Let (L_n), $n \in \mathbb{N}$, be a sequence of linear operators from E in C. Then, R_F-lim $L_n f = f$ for every $f \in E$ if and only if:*

(i) *(L_n), $n \in \mathbb{N}$, is an R_F-sequence;*

(ii) $R_F - \lim L_n 1 = 1$;

(iii) *for every $i = 1, 2, \ldots, p$,*

$$R_F - \lim \operatorname{Re} L_n f_i = f_i \quad \text{and} \quad R_F - \lim \operatorname{Re} L_n f_i^2 = f_i^2.$$

Proof. For every $x \in [a,b]$, we can define the function

$$f_x = \sum_{i=1}^{p} (f_i - f_i(x))^2.$$

It is very easy to check that $\{f_x\}$, $x \in [a, b]$, is a test family of functions of E that satisfies the conditions of Theorem 1.5. Thus the result immediately follows.

A particularly useful case is given by:

COROLLARY 4.2. *The algebraic polynomials are R_F-dense in R.*

Proof. Without loss of generality we can assume that $[a, b] = [0, 1]$. We can use the generalized p-dimensional Bernstein operators defined by

$$B_n f(x) = \sum_{i_1, \ldots, i_p = 0}^{n} \binom{n}{i_1} \cdots \binom{n}{i_p} f(i_1/n, \ldots, i_p/n)$$
$$\times x_1^{i_1}(1 - x_1)^{n-i_1} \cdots x_p^{i_p}(1 - x_p)^{n-i_p}$$

which form an R_F-sequence. As the finite set of functions considered in Corollary 4.1 we take here

$$f_i(x) = x_i, \qquad i = 1, 2, \ldots, p$$

and the statement follows immediately.

REFERENCES

1. W. DICKMEIS, H. MEVISSEN, R. J. NESSEL, AND E. VAN WICKEREN, Sequential convergence and approximation in the space of Riemann integrable functions, *J. Approx. Theory* **55** (1988), 65–85.
2. H. MEVISSEN, R. J. NESSEL, AND E. VAN WICKEREN, On the Riemann convergence of positive linear operators, preprint, Lehrstuhl A fur Mathematik Rheinisch-Westlische technische Hochschule Aachen, 332/*Me-Ne-Wi/IX*, 1986.
3. M. A. JIMENEZ POZO AND E. LOPEZ NUNEZ, Teorema cualitativo de tipo Korovkin en el espacio de las funciones Riemann integrables, *Cienc. Mat.* **11**, No. 2 (1990), 75–80.
4. E. LOPEZ NUNEZ, "Teoremas Cualitativos de Tipo Korovkin en Espacios de Funciones Riemann Integrables," Doctoral thesis, Havana University, 1989.
5. M. CAMPITI, A Korovkin-type theorem in the space of Riemann integrable functions, *Collect. Math.* **38** (1987), 199–228.
6. M. CAMPITI, Riemann sequential approximation of continuous functions, *Boll. Un. Mat. Ital.* B (7) **4** (1990), 143–154.
7. J. L. FERNANDEZ MUNIZ, Teoremas cualitativos de tipo Korovkin para sucesiones de operadores de clase R, *Cienc. Mat.* **3**, No. 1 (1982), 57–69.